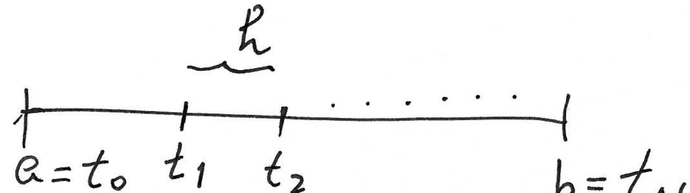
# 5.2 Euler's Method

Initial Value Problem (IVP)

$$\begin{cases} \dfrac{dy}{dt} = f(t,y), & a \le t \le b \quad (1.1) \\ y(a) = \alpha & \quad (1.2) \end{cases}$$

Mesh:




Equally distributed:

$$h = t_{i+1} - t_i = \frac{b-a}{N}$$

$N+1$ points : $t_0, t_1, \dots, t_N$

$N$ Subintervals : $(t_0, t_1), (t_1, t_2), \dots, (t_{N-1}, t_N)$

## Derivation of Euler Method.

Assume $y(t)$ is a unique soln. for (1)-(2).
and it has two conts. derivatives on $[a,b]$
Expanding $y(t)$ around $t = t_i$ using Taylor's thm.

$$y(t) = y(t_i) + (t-t_i)\,y'(t_i) + \frac{(t-t_i)^2}{2}\,y''(\xi_i)$$

for $\xi$ in $(t, t_i)$

In particular, for $i=0,1,2,\ldots,N-1$.

$$y(t_{i+1}) = y(t_i) + (\underbrace{t_{i+1}-t_i}_{=h})\,y'(t_i) + \frac{(\overbrace{t_{i+1}-t_i}^{=h})^2}{2}\,y''(\xi_i)$$

or
$$y(t_{i+1}) = y(t_i) + h\,y'(t_i) + \frac{h^2}{2}\,y''(\xi_i)$$

$$= y(t_i) + h\,f(t_i, y(t_i)) + \frac{h^2}{2}\,y''(\xi_i)$$

by calling $w_i \simeq y(t_i)$, $i=1,2,\ldots,N$.

we obtain the equations defining Euler's method neglecting the last term.

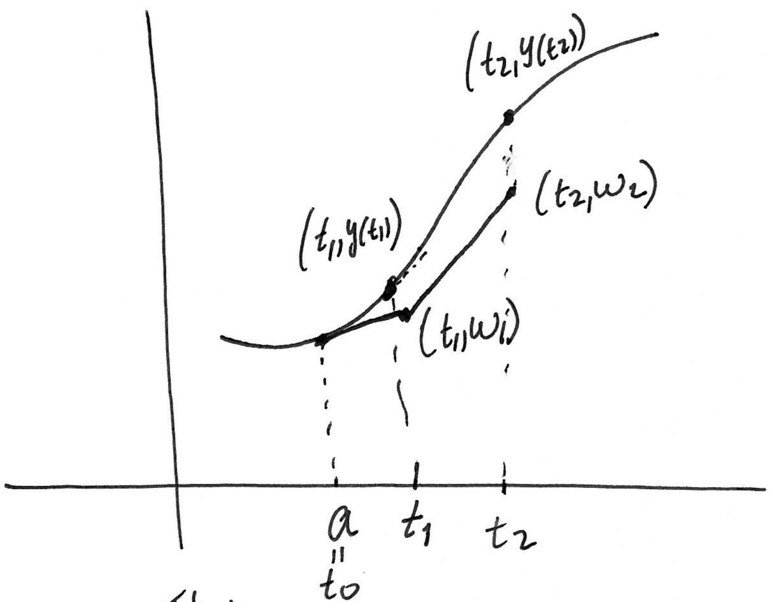
Difference equation
$$\begin{cases} w_{i+1} = w_i + h\,f(t_i, w_i) & (2.1) \\ w_0 = \alpha & (2.2 \end{cases}$$

for $i=0,1,\ldots,N-1$.

Show some MATLAB examples.

# Geometrical Interpretation of Euler's Method

$$f(t_i, w_i) \simeq y'(t_i) = f(t_i, y(t_i))$$

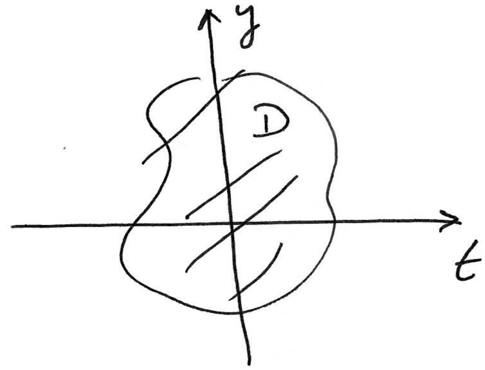



$$w_1 = \overset{\text{If exact } = y_0}{w_0} + (t_1 - t_0)\,\overset{= y'(t_0)}{f(t_0, w_0)}$$

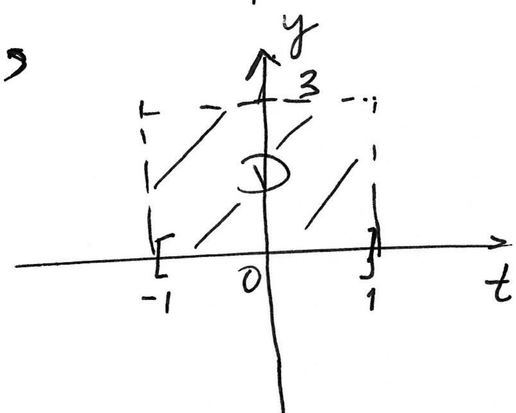
$$w_2 = \overset{\simeq y(t_1)}{w_1} + (t_2 - t_1)\, \overset{\simeq y'(t_1)}{f(t_1, w_1)}$$

# Lipschitz Condition

**Def.** Consider a set $D \subset \mathbb{R}^2$




For example,




$$D = \{(t,y) : -1 \le t \le 1, \quad 0 \le y \le 3\}$$

and

$$f : D \to \mathbb{R}$$
$$(t,y) \to f(t,y).$$

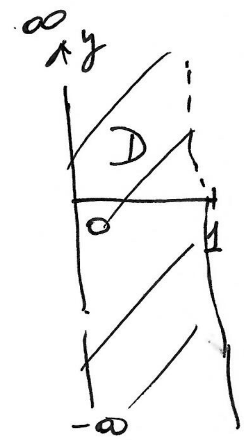
A function satisfies a <u>Lipschitz condition</u> in the variable $y$ on a set $D \subset \mathbb{R}^2$ if a constant $L$ exists with

$$|f(t,y_1)-f(t,y_2)| \le L|y_1-y_2|$$

for all $(t_1,y_1)$ and $(t_2,y_2)$ in $D$. And the constant $L$ is called a <u>Lipschitz constant.</u>

<u>Examples:</u>

i) $f(t,y) = y\cos t, \quad 0 \le t \le 1 \quad D=$




$$|f(t,y_1)-f(t,y_2)| =$$
$$= |y_1\cos t - y_2 \cos t|$$
$$= |y_1-y_2|\,|\cos t| \le |y_1-y_2|$$

So $\boxed{L=1}$ Lipschitz constant.

ii) $f(t,y) = \dfrac{1+y}{1+t}, \quad 0 \le t \le 1$

$$|f(t,y_1)-f(t,y_2)| = \left|\frac{\cancel{1}+y_1}{1+t} - \frac{\cancel{1}+y_2}{1+t}\right| \le \frac{1}{1+t}|y_1-y_2|$$
$$\overset{t=0}{\le} |y_1-y_2|$$

so, $\boxed{L=1}$

## Theorem 5.9

$D = \{(t,y) \mid a \le t \le b, \ -\infty < y < \infty\}$

i) $f$ is cont. in $D$.

ii) $f$ satisfies a Lipschitz condition with constant $L$ on $D$.

iii) $|y''(t)| \le M$, for all $t \in [a,b]$.

where $y(t)$ is the unique soln. of the IVP:

$$\begin{cases} y'(t) = f(t, y(t)), & a \le t \le b \\ y(a) = \alpha \end{cases}$$

Then,

if $w_0, w_1, .., w_N$ is the discrete approximation generated by Euler's method for $N+1$ grid points, then

$$|y(t_i) - w_i| \le \frac{hM}{2L}\left[e^{L(t_i - a)} - 1\right]$$

Proof. (thm 9)

Euler's equations for this IVP are

$$\begin{cases} w_{i+1} = w_i + h f(t_i, w_i) \\ w_0 = \alpha \end{cases}$$

then for $t = t_0 = a$

$y(t_0) = \alpha = w_0$ and $|y(t_0) - w_0| = 0 \leq \ldots$ trivial.

In general, if $t = t_i$, $i = 1, \ldots, N-1$

$$y(t_{i+1}) - w_{i+1} = y(t_i) + h f(t_i, y(t_i)) + \frac{h^2}{2} y''(\xi_i)$$
$$- \left(w_i + h f(t_i, w_i)\right)$$
$$= y(t_i) - w_i + h\left(f(t_i, y(t_i)) - f(t_i, w_i)\right) + \frac{h^2}{2} y''(\xi_i)$$

Then,

$$|y(t_{i+1}) - w_{i+1}| \leq |y(t_i) - w_i| + h\,|f(t_i, y(t_i)) - f(t_i, w_i)|$$
$$+ \frac{h^2}{2}|y''(\xi_i)|$$

Using the Lipschitz condition

$$|y(t_{i+1}) - w_{i+1}| \leq |y(t_i) - w_i| + hL\,|y(t_i) - w_i| + \frac{h^2}{2}\,|y''(\xi_i)|, \quad \xi_i \in (t_i, t_{i+1}).$$

Therefore,

$$|y(t_{i+1}) - w_{i+1}| \leq (1 + hL)\,|y(t_i) - w_i| + \frac{h^2}{2} M$$

Using two technical Lemmas 5.7 and 5.8 in Burden.

$$\leq \frac{hM}{L}\left(e^{L(t_{i+1} - a)L} - 1\right)$$

See application to Exercise 1) © in Burden in next page.

1) c) Find error bound for the IVP:

$$y' = 1 + y/t, \quad 1 \le t \le 2, \quad y(1) = 2, \quad h = 1/4$$

using Euler's method.

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We need to determine L and M.

<u>For L</u>:

$$|f(t,y_1) - f(t,y_2)| = \left| \frac{y_1}{t} - \frac{y_2}{t} \right|$$

$$= \frac{1}{t} |y_1 - y_2| \le |y_1 - y_2| \quad \text{for } t \in [1,2].$$

Then, $\boxed{L = 1}$

<u>For M</u>: Exact soln: $y(t) = t \ln t + 2t$

$$y'(t) = t/t + \ln t + 2 = 3 + \ln t$$

$$y''(t) = 1/t \Rightarrow |y''(t)| \le 1 \Rightarrow \boxed{M = 1}$$

Then,

$$\text{Error bound} = E_i = \frac{hM}{2L}\left(e^{(t_i - a)L} - 1\right) = \frac{1}{8}\left(e^{t_i - 1} - 1\right)$$

In particular, at $t_1 = 5/4$, $E_i = \frac{1}{8}\left(e^{1/4} - 1\right) = 0.0355$

Same as book